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ON GRAPHICAL PARTITIONS

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An integer partition $\{\lambda_1, \lambda_2, \dots, \lambda_v\}$ is said to be graphical if there exists a graph with degree sequence $\langle \lambda_i \rangle$. We give some results corcerning the problem of deciding whether or not almost all partitions of even integer are non-graphical. We also give asymptotic estimates for the number of partitions with given rank.

1. Introduction

In this paper a graph shall have unlabelled vertices, no loops and no multiple edges. The degree of a vertex shall be the number of edges incident with it. According to Harary [8, p14] the earliest result in graph theory, due to Euler, is that the sum of the degrees of a graph equals twice the number of edges. Let us define a graphical partition to be a partition whose parts are the degrees of the vertices of a graph, that is, whose parts are a degree sequence. In this paper we show that the number, $G(n)$, of graphical partitions of the even number n satisfies the following relations:

$$\overline{\lim}_{n\to\infty} \frac{G(n)}{p(n)} \le .4258 \tag{1}$$

$$\underline{\lim}_{n\to\infty} n^{1/2} \frac{G(n)}{p(n)} \ge \frac{\pi}{\sqrt{6}} \tag{2}$$

where $p(n)$ denotes the number of unrestricted partitions of n.

The rank of a partition, first defined by F. J. Dyson [5], equals the largest part minus the number of parts. The k-th rank, introduced by A. O. L. Atkin [2], equals the k-th largest part minus the number of summands greater than or equal to k. Results concerning the ranks of a partition are useful in our study of graphical partitions. We prove that the number of partitions of n with rank r, where

$$r = y6^{1/2}n^{1/2}/\pi + o(n^{1/2})$$

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is asymptotic to

$$\frac{p(n)}{n^{1/2}}\frac{\pi}{\sqrt{6}}/(1+e^{y})(1+e^{-y}). \tag{3}$$

Using additional results concerning ranks of partitions one may show that the number of partitions which satisfy the first k Erdős–Gallai conditions (see Theorem (2.1) below) is asymptotic to

$$c_k p(n), \quad c_k < c_{k-1} \tag{4}$$

where the c_k are positive constants. We do not give complete details since they are complicated and we cannot decide whether or not $G(n)/p(n)\to 0$ even with the results thereby obtained.

We show that almost all graphical partitions may be realized by a connected graph but that almost none of the graphical partitions may be realized by a planar graph. It follows from some results of E. M. Wright that the number of graphs with n edges divided by the number of graphical partitions of $2n$ is asymptotic to the number of connected graphs with n edges divided by the number of partitions of $2n$ whose parts are the degree sequence of a connected graph. It also follows that there is a graphical partition of $2n$ which is the degree sequence of $(n!)^{2-\varepsilon}$ connected graphs.

Eq. (1.2) has an interesting consequence in connection with the results of Szalay and Turán ([11], [12] and [13]) on the parts of partitions. They show that if λ_k denotes the k-th largest part of a partition then for a wide range of k

$$\lambda_k = \left(1+O\left(\frac{1}{\log n}\right)\right)\frac{\sqrt{6n}}{\pi}\log\left(\frac{1}{1-\exp\left(\frac{-\pi k}{\sqrt{6n}}\right)}\right)$$

holds uniformly for at least $(1-c\frac{\log n}{n^{5/4}})p(n)$ partitions. This result cannot be used with some condition that graphical partitions satisfy to show that most partitions are not graphical since it would rule out too many partitions in view eq. (1.2). It is nevertheless quite possible that this result will be essential in deciding whether or not $G(n)/p(n)\to 0$.

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2. Proofs

The proofs of both inequalities (1.1) and (1.2) start from Erdős–Gallai conditions [6] (see also [8, pp. 59–61] for a partition to be graphical.

Theorem 2.1. *(Erdős–Gallai). The sequence* $(d_1,d_2,\dots,d_s)$ *with* $d_1 \geq d_2 \geq \dots \geq d_s$ *is graphical if and only if* $\sum d_i$ *is even and*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{s} \min(k, d_i), \qquad \text{for} \quad 1 \leq k \leq s. \tag{5}$$

We first interpret these conditions in terms of the ranks of partitions. We may rewrite the right-hand side of (2.1) as

$$k(k-1) + \sum_{d_i=1}{}' 1 + \sum_{d_i=2}{}' 2 + \dots + \sum_{d_i \geq k}{}' k$$

where $\sum'$ denotes summation over $i \geq k+1$. This is

$$\geq k(k-1) + \left(-k + \sum_{d_i \geq 1} 1\right) + \left(-k + \sum_{d_i \geq 2} 1\right) + \dots + \left(-k + \sum_{d_i \geq k} 1\right)$$
$$= -k + \sum_{d_i \geq 1} 1 + \sum_{d_i \geq 2} 1 + \dots + \sum_{d_i \geq k} 1.$$

Thus if the sum of the first k ranks is $\leq -k$ the the k-th Erdős–Gallai condition is satisfied. In particular if eacgh successive rank is negative the partition is graphical. The following Theorem proved for all odd M by Andrews [1] and extended to all M by Bressoud [4] enables one to count such partitions:

Theorem 2.2. *(Andrews–Bressoud): Given positive integral* M *and integral* $r, 0 < r < m/2$, *let* $A_{M,r}(n)$ *denote the number of partitions of* n *into parts* $\not\equiv 0, \pm$ (mod M). Let $B_{M,r}(n)$ denote the number of partitions of n whose successive ranks lie in the interval $[-r+2, M-r-2]$. Then $A_{M,r}(n) = B_{M,r}(n)$ for all n.

If we let $r=1$, $M=n+2$ then $B_{M,r}(n)$ equals the number of partitions of n with positive ranks which in turn equals the number of partitions with negative ranks (by interchanging rows and columns in the Ferrar's diagram of the partitions). $A_{M,r}(n)$ equals the number of partitions of n with no part equal to one and we have.

Corollary 2.1. *The number of partitions of* n *with negative successive ranks is* $p(n) - p(n-1)$

Eq. (1.2) follows at once since from Roth and Szekeres [10]

$$p(n) - p(n-1) \sim \frac{\pi}{\sqrt{6n}} p(n).$$

Remark. Nash-Williams [9] shows that if K is defined to be the greatest value of k for which $a_k \geq k$ then a partition is graphical if and only if $r_1 + r_2 + \dots + r_k \leq -k$ for $k \leq K$ It would be quite useful for the discussion of $G(n)$ to enumerate partitions satisfying this rank condition.

We now prove that

$$\overline{\lim}_{n\to\infty}\frac{G(n)}{p(n)} \le .5.$$

It if sufficient to consider the $k=1$ Erdős–Gallai condition. Let $N(<0,n)$ denote the number of partitions of N with rank <0 and $N(0,m)$ the number with rank $=0$. Interchanging the rows and columns of the Ferrar's diagram gives a 1–1 correspondence between the partitions with rank m and rank $-m$ so $N(-m,n)=N(m,n)$, Hence $2N(<0,n)=p(n)-N(0,n)$ and $N(<0,n)\sim p(n)/2$ iff $N(0,n)=o(p(n))$.

Lemma 2.1.

$$N(0,n)=o(p(n)).$$

Proof. Assume that $N(0,n)\ge cp(n)$ for some positive constant $c>0$. The number of partitions of n that have at least k ones is $p(n-k)$. Thus for any fixed k almost all partitions of n have k ones and hence by our assumption almost all partitions with rank zero have $\ge 2l$ ones. For these we can construct l distinct partitions of nonzero rank by replacing some or all of the l pairs of ones by twos. The partitions of nonzero rank obtained in this way from the partitions of zero rank are all distinct and if we chose $k=2[1/c]+2$ we obtain a total of $cp(n)(2[1/c]+2)>p(n)$ partitions which is impossible.

Before proving (1.3) and (1.4), we establish the claims in the last paragraph of Section 1. To facilitate this process, we first make a general observation concerning graphical partitions. By a result of Szalay and Turán [[13], Lemma 4], with the exception of at most $0(n^{-2}p(n))$ partitions of n, the number of summands v satisfies

$$v\le\frac{5\sqrt{6}}{2\pi}\sqrt{n}\log n.$$

In view of (1.2), the above inequality holds for almost all graphical partitions $n=2e$.

Call a partition c-graphical if it is the degree sequence of a connected graph. It is known that a graphical partition on n with $v\ge 3$ summands is c-graphical iff $n\ge 2(v-1)$ [8, p. 63]. Of course, almost every graphical partition has $v\ge 3$ summands. In view of the preceding observation, almost all graphical partitions are c-graphical.

Call a graphical partition p-graphical if it is the degree sequence of a planar graph. It is well-known fact that $e\le 3v-6$ for any planar graph. Thus if $d_1,d_2,\dots,d_v$ is a p-graphical partition of n, then $n\le 6v-12$. It follows that almost all graphical partitions fail to be p-graphical.

Sir Edward Wright communicated the following observations to us. Let T denote the set of connected graphs in n edges. A graph on n edges has at most $2n$ vertices so let U denote the set of graphs with n edges and $2n$ vertices. Let Q denote the set of graphs with n edges and no isolated vertices. There is natural 1–1 correspondence between Q and U (simply remove this isolated vertices from the graphs of U) so $|U|=|Q|$. By Theorem 1 of Wright [14] almost all graphs of U consist of one connected component and isolated vertices hence $|t|\sim|Q|=|U|$ as $n\to\infty$. All the remaining statements in the second-last paragraph of Section 1 follow from this fact and Theorem 5 of Wright [15] (which gives an asymptotic formula for $|T|$).

We now improve (1.1), prove (1.3) and sketch the proof of (1.4) which requires us to investigate the ranks of partitions. Let $P_n(d_1, v)$ denote the number of partitions of n with largest summand d_1 and number of summands v. Then the number of partitions of n with rank r is given by

$$N(r,n) = \sum_{d_1} P_n(d_1, d_1 - r).$$

By an easy modification of the arguments of Auluck, Chowla and Gupta [3] (which are based on the techniques of Erdős and Lehner [7]), one finds that with

$$d_1 = \frac{\sqrt{6n}}{2\pi}\log n + \frac{x\sqrt{6n}}{\pi} + o(\sqrt{n}),$$

and

$$v = \frac{\sqrt{6n}}{2\pi}\log n + \frac{y\sqrt{6n}}{\pi} + o(\sqrt{n}),$$

the asymptotic formula

$$P_n(d_1, v) \sim \frac{p(n)}{n} F(x)F(y)$$

holds, where

$$F(x) = \exp\left(-x - \frac{\sqrt{6}}{\pi}e^{-x}\right).$$

It follows by routine methods that

$$N\left(\left\lfloor \frac{t\sqrt{6n}}{\pi} \right\rfloor, n\right) \sim \frac{p(n)\pi}{n}\frac{\sqrt{6n}}{\pi}\int_{-\infty}^{\infty} F(x)F(x-t)dx \quad (n \to \infty),$$

and

$$\int_{-\infty}^{\infty} F(x)F(x-t)dx = \frac{\pi^2 e^t}{6(1+e^t)^2}.$$

Thus we obtain

$$N\left(\left\lfloor \frac{t\sqrt{6n}}{\pi}, n \right\rfloor\right) \sim \frac{p(n)\pi e^t}{\sqrt{6n}(1+e^t)^2} \quad (n \to \infty),$$

which is (1.3).

We now prove (1.1) using this type of argument. Let us estimate the number of partitions satisfying the first 2 Erdős–Gallai conditions. For almost all partitions of n it's true that $d_1 = c^{-1}n^{1/2}\log n + 2c^{-1}xn^{1/2} + o(n^{1/2})$ and $s_1 = c^{-1}n^{1/2}\log n + 2c^{-1}(x+y)n^{1/2} + o(n^{1/2})$. If a partition satisfies the first Erdős–Gallai condition then $y \geq 0$. Having fixed d_1 and s_1 we may construct a partition of n with given d_1 from a partition of $n - d_1 - s_1$ by adding d_1 dots as the first row and s_1 dots to the bottom column of the Ferrar's diagram of the partition of $n-d_1-s_1$ iff the largest summand d_2 satisfies $d_2 \le d_1$ and the number of summands s_2 satisfies $s_2 \le s_1$. It easily seen using (2.2) that the number of partitions of $n-22c^{-1}n^{1/2}\log n - 2c^{-1}(2x_1+y_1)n^{1/2}+o(n^{1/2})$ with

$$d_2 = c^{-1}n^{1/2}\log n + x_2\frac{\sqrt{6}}{\pi}n^{1/2} + o(n^{1/2}), \quad x_2 \le x_1$$

and

$$s_2 = c^{-1}n^{1/2}\log n + (x_2+y_2)\frac{\sqrt{6}}{\pi}n^{1/2} + o(n^{1/2}), \quad x_2+y_2 \le x_1+y_1$$

is

$$\sim p(n-2c^{-1}n^{1/2}\log n - 2c^{-1}(2x_1+y_1)n^{1/2}+o(n^{1/2}))\cdot e^{-2x_2-y_2-\frac{\sqrt{6}}{\pi}e^{-x_2}(1+e^{-y_2})}$$

and from the Hardy–Ramanujan asymptotic formula for $p(n)$ this is

$$\sim \frac{p(n)}{n^2}\exp(-2x_1-y_1-2x_2-y_2-\frac{\sqrt{6}}{\pi}e^{-x_2}(1+e^{-y_2}))$$

Hence the number of partitions satisfying E.–G. 1) and 2) is

$$\sim p(n)\left(\frac{\sqrt{6}}{\pi}\right)^4 \int\limits_{x_1=-\infty}^{\infty}\int\limits_{y_1=0}^{\infty}\int\limits_{x_2=-\infty}^{x_1}\int\limits_{y_2=-y_1}^{x_1+y_1-x_2} e^{-2x_1-2x_2-y_1-y_2-\frac{\sqrt{6}}{\pi}e^{-x_2}\cdot(1+e^{-y_2})}$$

$$= p(n)\cdot 45 = c_2 p(n), \qquad \text{giving (1.1).}$$

(If the range of integration of y_2 is relaxed we obtain $c_1 = .5$). The reader will perhaps verify that there is no difficulty in principle of extending this to the first k E.–G. conditions. (We obtained $c_3 = .42$.) Perhaps it is worth noting that from (1.3) the number of partitions satisfying E.–G. 1) is

$$\sim p(n)\int\limits_{-\infty}^{0}\frac{dy}{(e^y+1)(1+e^{-y})} = \frac{p(n)}{2}.$$

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